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Plane Regions Determined by Complex Moments

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1. INTRODUCTION

Let B designate a bounded region assumed to lie in the complex $z = x + iy$ plane. The complex numbers

$$\tau_n = \iint_B z^n \, dx\, dy, \qquad n = 0, 1, 2,..., \tag{1.1}$$

will be called the *complex moments* of the region B. To what extent is B determined uniquely by the numbers τ_n or by a subset of them? If there is no uniqueness, how might the family $\mathscr{B}$, from which competing regions B have been extracted, be restricted so that uniqueness results? This is obviously a question about the completeness (in some sense) of the complex powers z^n, $n = 0, 1,...$.

In dealing with two dimensional regions, one normally deals with the real moments

$$\mu_{m,n}(B) = \mu_{m,n} = \iint_B x^m y^n \, dx\, dy, \qquad m, n = 0, 1,.... \tag{1.2}$$

Introducing the numbers $\tau_{m,n}$ by means of

$$\tau_{m,n} = \iint_B z^m \bar{z}^n \, dx\, dy, \qquad m, n = 0, 1,..., \bar{z} = x - iy, \tag{1.3}$$

one has

$$\tau_{m,n} = \sum_{\substack{j=0\\k=0}}^{m,n} (i)^{m-j} (i)^{n-k} \binom{m}{j}\binom{n}{k} \mu_{j+k,m+n-j-k} \qquad (i = (-1)^{1/2}) \tag{1.4}$$

or, in the reverse direction,

$$\mu_{m,n} = (-i)^n \, 2^{-m-n} \sum_{\substack{j=0\\k=0}}^{m,n} \binom{m}{j}\binom{n}{k} \tau_{j+k,m+n-j-k} \,. \tag{1.5}$$

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Thus, given all the moments $\mu_{m,n}$, we may determine all the moments $\tau_{m,n}$ and vice versa. The matrix

$$M = (\tau_{m,n}) \tag{1.6}$$

cannot be arbitrary. It is, for example, Hermitian ($M = M^*$), so that only the upper triangle is necessary to reconstruct all the moments $\tau_{m,n}$. Then, again, its elements must satisfy the Schwarz Inequalities

$$|\tau_{m,n}|^2 \leqslant \tau_{m,m}\tau_{n,n}. \tag{1.7}$$

With respect to the moments (1.2) or (1.3), the uniqueness situation is governed by the following typical theorems that are based upon the real powers $x^m y^n$, $m, n = 0, 1, 2, \ldots$.

THEOREM. *Let $\mathscr{B}$ designate the family of plane sets which are measurable and which lie in a fixed disc C. Let L(C) designate the class of functions that are defined on C and are integrable. Let $f_n(x, y)$, $n = 1, 2, \ldots, \in L(C)$. Given B and $D \in \mathscr{B}$. A necessary and sufficient condition that*

$$\iint_B f_n(x, y)\,dx\,dy = \iint_D f_n(x, y)\,dx\,dy, \qquad n = 1, 2, \ldots, \tag{1.8}$$

imply $B = D$ a.e. is that the sequence $f_n(x, y)$ be complete in $L(C)$.

THEOREM. *Let B and D be bounded open sets which possess exterior points in the neighborhood of any boundary point. Then,*

$$\iint_B x^m y^n\,dx\,dy = \iint_D x^m y^n\,dx\,dy, \qquad m, n = 0, 1, \ldots, \tag{1.9}$$

implies $B = D$.

For proofs of these statements see, e.g., [3 pp. 198–200].

Now, the basic question we have raised here is what can be said on the basis of knowledge only of the numbers

$$\tau_n = \tau_{n,0} = \iint_B z^n\,dx\,dy, \qquad n = 0, 1, \ldots. \tag{1.10}$$

We cannot obtain all the values (1.2) from a knowledge only of (1.10)—at least, not through (1.5). (For example, we cannot express $x^2 + y^2 = z\bar{z}$ as a polynomial in $1, z, z^2$.) We are dealing with a subset of the matrix M, and hence with a problem of Müntz type.

In the present paper, we make the assumption that B is restricted to be one of the simplest of elementary figures, namely, a triangle. We shall prove that a triangle T is uniquely determined by its first four complex moments.

2. Complex Moments of a Triangle

For three arbitrary complex numbers z_1, z_2, z_3, introduce the three elementary symmetric functions

$$\begin{aligned} \text{sum} = s &= z_1 + z_2 + z_3, \\ t &= z_1z_2 + z_2z_3 + z_3z_1. \\ \text{product} = p &= z_1z_2z_3. \end{aligned} \tag{2.1}$$

Furthermore, for n = integer, set

$$G_n = \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1{}^n & z_2{}^n & z_3{}^n \end{vmatrix}. \tag{2.2}$$

Note that G_2 is the Vandermonde of z_1, z_2, z_3 and

$$G_2 = (z_3 - z_1)(z_3 - z_2)(z_2 - z_1).$$

Lemma 1.

$$\frac{G_3}{G_2} = s, \qquad \frac{G_4}{G_2} = s^2 - t, \qquad \frac{G_5}{G_2} = s^3 - 2st + 7p. \tag{2.3}$$

Proof. Computation.

Note that G_n/G_2 is a symmetric polynomial in z_1, z_2, z_3, so that by the fundamental theorem of such polynomials, it must be a polynomial in s, t, and p. Lemma 1 gives the explicit representation for $n = 3, 4, 5$.

Let T designate a nondegenerate triangle whose vertices in any order are z_1, z_2, z_3. Let $A = A(T)$ designate the (positive) area of T.

Lemma 2. *Let $f(x)$ be any analytic function which is regular in T and continuous in the closure of T. Then*

$$\frac{1}{2A} \iint_T f''(z)\, dx\, dy = \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ f(z_1) & f(z_2) & f(z_3) \end{vmatrix} \div G_2. \tag{2.4}$$

Proof. See [2, p. 128] and also the references cited there. We have called (2.4) the *Motzkin–Schoenberg–Grunsky formula.*

Corollary 1. *For integer $n \geqslant 2$,*

$$(n(n-1)/2A) \iint_T z^{n-2}\, dx\, dy = G_n/G_2. \tag{2.5}$$

COROLLARY 2.

$$\tau_1 = \iint_T z\,dx\,dy = \frac{A}{3}\frac{G_3}{G_2} = \frac{A}{3}s,$$

$$\tau_2 = \iint_T z^2\,dx\,dy = \frac{A}{6}\frac{G_4}{G_2} = \frac{A}{6}(s^2 - t), \tag{2.6}$$

$$\tau_3 = \iint_T z^3\,dx\,dy = \frac{A}{10}\frac{G_5}{G_2} = \frac{A}{10}(s^3 - 2st + 7p).$$

Proof. Corollary 1 and Lemma 1.

COROLLARY 3.

$$\begin{aligned} s &= 3\tau_1 \div \tau_0, \\ t &= (9\tau_1^2 - 6\tau_0\tau_2) \div \tau_0^2, \\ p &= (10\tau_0^2\tau_3 + 27\tau_1^3 - 36\tau_0\tau_1\tau_2) \div \tau_0^3. \end{aligned} \tag{2.7}$$

Proof. $A = \tau_0$. Now solve (2.5) successively.

THEOREM. *A triangle T is determined uniquely by its first four complex moments $\tau_k = \iint_T z^k\,dx\,dy$, $k = 0, 1, 2, 3$.*

Proof. The first four moments determine s, t, p uniquely through (2.7). Now since $\phi(z) = (z - z_1)(z - z_2)(z - z_3) = z^3 - sz^2 + tz - p$, the polynomial $\phi(z)$ is determined uniquely. Hence also the roots z_1, z_2, z_3, up to a permutation.

COROLLARY. *A triangle in the x, y plane is determined uniquely by the seven real numbers*

$$A(T) = \iint_T dx\,dy, \quad \iint_T x\,dx\,dy, \quad \iint_T y\,dx\,dy, \quad \iint_T (x^2 - y^2)\,dx\,dy,$$

$$\iint_T xy\,dx\,dy, \quad \iint_T (x^3 - 3xy^2)\,dx\,dy, \quad \iint_T (3x^2y - y^3)\,dx\,dy.$$

Proof. Split τ_k, $k = 0, 1, 2, 3$ into real and imaginary parts.

It should be observed that the identities (2.7) coupled with $\phi(z)$ provide an algorithm for constructing T from given values τ_0, τ_1, τ_2, τ_3. Note further that four arbitrary numbers are not necessarily the first four moments of a T. *Example*: $\tau_0 = A > 0$, $\tau_1 = 0$, $\tau_2 = \tau_3 = 0$.

3. SOME COUNTEREXAMPLES

(a) *Sets of measure* 0. If B and D differ by a plane set of measure 0, then, of course, $\mu_{m,n}(B) = \mu_{m,n}(D)$ and $\tau_{m,n}(B) = \tau_{m,n}(D)$. This is mentioned because B and D may be essentially different as far as conformal mapping is concerned. For example: B = unit disc., D = unit disc with $0 \leqslant x \leqslant 1$ removed.

(b) *Connectivity.* Let B and D be two distinct annuli centered at $z = 0$ and such that area(B) = area(D). Then, as an easy computation shows, $\tau_n(B) = \tau_n(D)$, $n = 0, 1, 2, \ldots$.

(c) *Algebraic dependence.* It might be thought, in view of the above theorem, that any four numbers $\tau_{m,n}$ will determine a triangle uniquely. But this is not so. For example, the four numbers $\tau_{0,0}$, $\tau_{1,0}$, $\tau_{2,0}$, $\tau_{1,1}$ do not serve to determine T uniquely.

To show this, let T_1 and T_2 be any two equilateral triangles whose area is a fixed constant A and whose center of gravity is at $z = 0$. Therefore $\tau_{0,0}(T_1) = \tau_{0,0}(T_2) = A$. If $z_{\text{c.g.}}$ designates the center of gravity of a triangle T, then $z_{\text{c.g.}} = (1/A) \iint z \, dx \, dy = s/3$. Therefore, $\tau_{1,0}(T_1) = \tau_{1,0}(T_2) = 0$. If T is equilateral, then its vertices can be represented as $z_1 = z^*$, $z_2 = z^*\omega$, $z_3 = z^*\omega^2$ with $\omega^3 = 1$ and $| z^* | = \sigma$. Therefore $t = z_1z_2 + z_2z_3 + z_3z_1 = z^{*2}(\omega + 1 + \omega^2) = 0$. Thus, $t(T_1) = t(T_2) = 0$ so that by (2.6), $\tau_{2,0}(T_1) = \tau_{2,0}(T_2)$.

Finally, for any triangle T,

$$\iint_T z\bar{z} \, dx \, dy = (A/12)(| z_1 |^2 + | z_2 |^2 + | z_3 |^2 + | z_1 + z_2 + z_3 |^2). \quad (3.1)$$

(See [1, p. 575].) Therefore,

$$\begin{aligned} \tau_{1,1}(T_1) &= \iint_{T_1} z\bar{z} \, dx \, dy = (A/12)(| z_1 |^2 + | z_2 |^2 + | z_3 |^2) = A\sigma^2/4 \\ &= \tau_{1,1}(T_2). \end{aligned}$$

What is involved here, of course, are moment *invariants*. For example, if $m + p = n + q$ then the quantities $\tau_{m,n}\tau_{p,q}$ are invariant under rotations about the origin. The question of invariance under certain groups is of importance in pattern recognition.

In a paper which follows, we plan to apply the ideas here to the problem of the approximation of one region by another.

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